Announcements

1) Bonus on exam 1 is 10 points extra credit; due in one week

2) HW #3 to appear later this week, due Thursday next week

Recall: imaginary solutions to $r^2 = \alpha$ when $\alpha < 0$.

Then solutions for r will "purely imaginary" of the form $r = \pm i\sqrt{-\alpha}$.

We'd still like real-valued solutions, though!

Remember the following MacLaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Relate these functions

by computing

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}$$

$i^0 = 1, \ i^1 = i, \ i^2 = -1,$

$i^3 = -i, \ i^4 = 1, \ldots -$

Split the sum into even and odd exponents:

$$\sum_{n=0}^{\infty} \frac{i^n x^n}{n!} = \sum_{n \text{ even}} \frac{i^n x^n}{n!} + \sum_{n \text{ odd}} \frac{i^n x^n}{n!}$$

Write an even number as

$n = 2k$, $k = 0, 1, 2, \ldots$

odd number as $n = 2k+1$.

The sum becomes

$$\sum_{k=0}^{\infty}\frac{i^{2k}x^{2k}}{(2k)!} + \underbrace{\sum_{k=0}^{\infty}\frac{i^{2k+1}x^{2k+1}}{(2k+1)!}}_{= i\sum_{k=0}^{\infty}\frac{i^{2k}x^{2k+1}}{(2k+1)!}}$$

$$i^{2k} = (i^2)^k = (-1)^k$$

We then get

$$\boxed{e^{ix} =} \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\boxed{= \cos(x) + i\sin(x)}.$$

(x is a real number).

Consequence:

If $e^{ix} = \cos(x) + i\sin(x)$,

then $e^{-ix} = \cos(-x) + i\sin(-x)$

$$= \underbrace{\cos(x)}_{\text{cos even}} - \underbrace{i\sin(x)}_{\text{sin odd}}$$

So $e^{ix} + e^{-ix} = 2\cos(x)$, and

$$\boxed{\cos(x) = \frac{e^{ix} + e^{-ix}}{2}}$$

Similarly,

$$\boxed{\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}}$$